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PREFACE

Greetings and welcome to the first volume of Proceedings FYP CS249—
It is my pleasure to welcome all of supervisors and FYP group ...

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d'Alembert Formula For Wave Equation

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Abstract

The title of this project is d'Alembert Formula For Wave Equation. The purpose of this project is to solve the wave equation problem by using d'Alembert formula and show that the wave is traveling towards opposite directions. First, d'Alembert formula is derived from the wave equation. Then, a few wave problems have been chosen and solved. Various graphs have been plotted by using Maple Software, which show that it is proven the wave is traveling towards two opposite directions. After all of the three steps of methodology have been done, the results achieved is as wanted. The results have shown that the d'Alembert formula can be used to solve wave equation problems. This formula also can be used to prove that the wave is travelling in two opposite directions.

Keywords: ODE, d'Alembert,

1 Introduction

In physics, the description of waves such as light waves, water waves and sound waves which is an important second-order linear partial differential equation can be described by the wave equation.

Historically, Jean le Rond d'Alembert, Daniel Bernoulli ,Joseph-Louis Lagrange and Leonhard Euler have been investigated about the problem of a vibrating spring (Cannon & Dostrovsky, 1983). D'Alembert had found the one-dimensional wave equation in 1746 and after ten years, Euler found the three-dimensional wave equation (Spesier, 2008).

This project involves one-dimensional wave equation which the standing wave is produced from two waves with the same amplitude, wavelength, and frequency travel in opposite directions.
The wave equation for u is

$$u_{tt} = a^2 u_{xx} \tag{1}$$

where a^2 is a fixed constant.

The solutions of the equations describe the propagation of disturbances out of the region at fixed speed. The constant a^2 is denoted as the propagation speed of wave. It is a linear equation. The superposition principle which is the solutions is again a solution or the sum of any two functions is thus applied in this project.

A physical solution cannot be specified by the wave equation alone. The problem is set with conditions, for instance the initial conditions, that described the amplitude and phase of wave, a unique solution is obtained. In this project, we are going to focus our study about wave equation, Eq.(1) itself without any modification.

The wave equation with initial conditions

$$u(x,0) = f(x) \tag{2}$$

$$u_t(x,0) = g(x) \tag{3}$$

and with boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{4}$$

According to Wazwaz (1998), d'Alembert and others handled this problem by using the oldest systematic method of the Separation of Variables. Another method is by using specific changes of variables from the solution $u(x,t)$ of the wave Eq. (1), d'Alembert had been formally derived from the wave equation in the form

$$u_{tt} - a^2 u_{xx} = 0 \tag{5}$$

This project is made to approach the wave Eq. (5) with the initial conditions (2) and (3) effectively by using a suitable decomposition method. The approach will be in the form of several examples. This is to confirm the belief of improvements that expected to be achieved by using this technique.

This formula is important in many applications and describes many physical phenomena for example, sound waves, ocean waves, compression wave, transverse wave, mechanical and electromagnetic waves.

The d'Alembert Formula, which is a general method for the solution of the one dimensional homogeneous wave equation is going to be derived. By using this formula we are going to prove that the wave equation is a superposition of two standing waves that propagate towards opposite directions.

2 Literature Review

Wave is said to be a disturbance that is propagating through space. The propagation of waves such as longitudinal wave, transverse wave, surface wave, and electromagnetic or mechanical wave can be described by wave equation. It usually transferring energy, as wave is said to be the energy or a disturbance propagating energy through a space or medium.

Contradict to d'Alembert who thought that the general solution of the problem had been found by him, Euler said that the initial position of the string is given by a single function has no physical reason required. The initial condition and the shape of the travelling wave have been operated by d'Alembert. (D'Alembert et al., 1949).

According to Davis et al. (1995),even if it is not given by a function, Euler argued that any graph, should be admitted as a possible initial position of the string. This physical reasoning by Euler could not be accepted by d'Alembert. In 1755, Daniel Bernoulli had joined the argument between d'Alembert and Euler with his paper.With the joining of Bernoulli, "standard waves", another form of solution for the vibrating string, have been found (D'Alembert et al., 1949).

A standing wave which is also known as a stationary wave, is a motion of the string.There are fixed nodes which are stationary and the string moves up and down simultaneously between the nodes of each segment. (Davis et al., 1995). In this case, which involve one-dimensional wave equation, two waves that travels in opposite directions will occur and produce a standing wave.

The statement of Bernoulli had been rejected by Euler. This is because the standing wave solution for a special case had be found by Euler already. (Davis et al., 1995). The solution by Bernoulli also had been rejected by d'Alembert.

The method from Euler was far more useful than d'Alembert's, but actually Bernoulli's method was the one that closer to the truth (Darrigol, 2007). The issue of the vibrating spring had been debated throughout 1760s and 1770s. The debate also joined by Laplace in 1779 and he was agreed with d'Alembert. (Kline, 1981).

According to Zill & Wright (2005), the one dimensional wave equation can be solved by using Separation of Variables and Fourier Series. While according to Kirkwood (2013), by using Laplace Transform, wave equation problems can be solved.

According to J.Farlow (1982) the problem has no boundaries and has been solved by d'Alembert in about 1750. d'Alembert Formula is also said to be an easy way to compute the wave equation problem when the initial conditions are given.

There are various ways in getting the d'Alembert Formula. According to Biazar et al. (2011), by using variational iteration and homotopy perturbation method the d'Alembert formula can be obtained. After they used both methods, the variational iteration and homotopy perturbation are certainly can be used to get the d'Alembert Formula. Jeffrey (2003) and according to Wazwaz (2009) the development of the d'Alembert Formula in the easiest way had been shown by using the techniques of simultaneous equations, factorization and simple integration.

When the general formula which is $u(x,t) = F(x+at) + G(x-at)$ with given initial condition is applied, the final form of the d'Alembert Formula is

$$u(x,t) = \frac{1}{2}[f(x-at) + f(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} G(s)ds$$

According to Erikkson et al. (1996), the initial conditions gives influence to the solution $u(x,t)$ at a given time t and they are two propagating waves moving to the opposite sides that is towards the left and the other one moves towards the right.

3 Methodology

3.1 STEP 1 : Derivation of d'Alembert Formula

Derive the D'Alembert Formula from wave equation

$$u_{tt} - a^2 u_{xx} = 0 \tag{6}$$
$$u(x,0) = f(x) \tag{7}$$
$$u_t(x,0) = g(x) \tag{8}$$

the general solution is

$$u(x,t) = F(x+at) + G(x-at) \tag{9}$$

and when initial condition is applied,

$$F(x) = \frac{1}{2}[f(x) + \frac{1}{a}\int g(s)ds] + A \tag{10}$$
$$G(x) = \frac{1}{2}[f(x) - \frac{1}{a}\int g(s)ds] + B \tag{11}$$

since $f(x) = F(x) + G(x)$ implies $A + B = 0$
Thus,

$$u(x,t) = F(x+at) + G(x-at) \tag{12}$$
$$F(x+at) + G(x-at) \tag{13}$$
$$= \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(s)ds + \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(s)ds \tag{14}$$
$$= \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds \tag{15}$$

3.2 STEP 2 : Use d'Alembert Formula to solve wave problems

The wave equation is solved by using d'Alembert formula.

$$u_{tt} - a^2 u_{xx} = 0$$

By using the D'Alembert formula $\frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds$ these problems are solved:

3.2.1 Problem 1

$$u(x,0) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$
$$u_t(x,0) = 0$$

3.2.2 Problem 2

$$u(x,0) = 0$$
$$u_t(x,0) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

3.2.3 Problem 3

$$u(x,0) = e^{-x^2}$$
$$u_t(x,0) = 0$$

3.2.4 Problem 4

$$u(x,0) = cos(x)$$
$$u_t(x,0) = 0$$

3.2.5 Problem 5

$$u(x,0) = sin(x)$$
$$u_t(x,0) = 1$$

3.2.6 Problem 6

$$u(x,0) = 0$$
$$u_t(x,0) = sin(2x)$$

3.3 STEP 3 : Show that there are two travelling waves

D'Alembert formula shows the wave equation is a superposition of two travelling waves that move towards opposite sides.

The solutions achieved after solving Problems 1, 2, 3, 4, 5 and 6 using d'Alembert solution. Maple Software have been used to plot the graphs of the solutions. All the graphs are included in the implementation. It is proven that the solutions show the wave is a travelling waves moving towards opposite directions.

4 Results And Discussion

Table below indicates the solutions obtained from the wave problems that have been solved from the Implementation Section. Assume that $a = 1$.

Table 1: Problems and its Solutions

Problem	**Solution**
$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}, g(x) = 0$	1
$f(x) = 0, g(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$	t
$f(x) = e^{-x^2}, g(x) = 0$	$\frac{1}{2}[e^{-(x+t)^2} + e^{-(x-t)^2}]$
$f(x) = cos(x), g(x) = 0$	$cos(x)cos(t)$
$f(x) = sin(x), g(x) = 1$	$sin(x)cos(t) + t$
$f(x) = 0, g(x) = sin(2x)$	$\frac{1}{2}[sin(2x)sin(2t)]$

4.1 Problem 1

The functions for the first problem are
$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$ and $g(x) = 0$.
The graph of the following problem has been plotted as per below.

shows the summation of two functions, $f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$ and $g(x) = 0$ solved by using d'Alembert equation are moving towards opposite directions. From the figure, it can be seen that, as time increases, the curve of functions show the following changes :

4.2 Problem 2

The functions for the second problem are $f(x) = 0$ and $g(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$.

4.3 Problem 3

The initial pulse is breaking into the superposition of two pulse of half the height which one pulse is moving to the left and the other one is moving to the right.

4.4 Problem 4

This is a standing wave even it certainly does not look offhand like a wave that is moving to the left and to the right, but that is what it is actually. By using

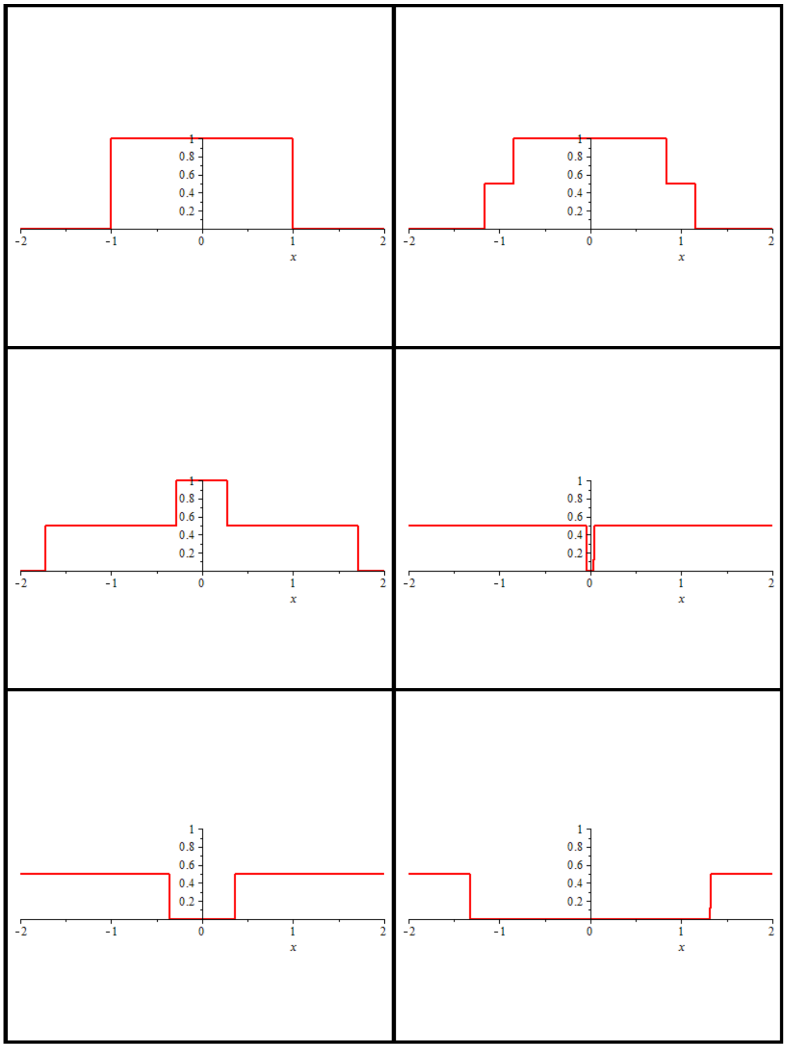

Figure 1: Addition of two functions, $f(x)$ and $g(x)$ stated above, as t is increasing

the trigonometry identities, the solution is $cos(t)cos(x)$.The multiplication of $cos(t)$ and $cos(x)$ shows that It is the $cos(x)$ that is moving to the left and right while $cos(t)$ moves to the up and down.This makes it oscillates back and forth.

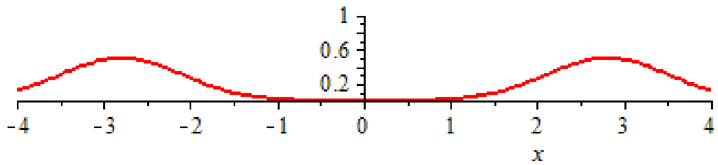

Figure 2: The wave equation solved from problem 3.

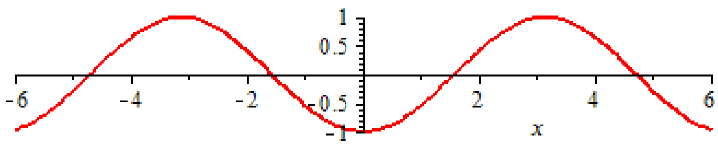

Figure 3: The wave equation solved from problem 4.

4.5 Problem 5

The solution for this graph is $sin(x)cos(t) + t$. The figure 4.5 shows that the pulse that moves to the left and right is $sin(x)$ while $cos(t)$ making the pulse to moves up and down. Since this is the summation of $sin(x)cos(t) + t$,it made the pulse to forever moves in upward direction.

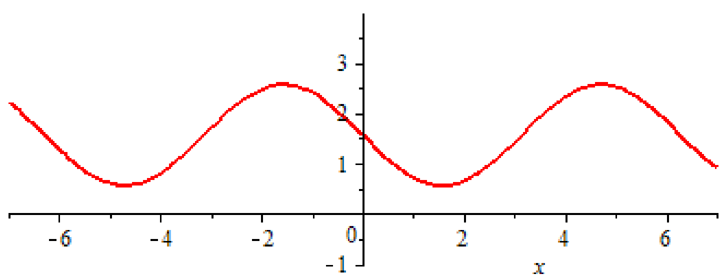

Figure 4: The wave equation solved from problem 5.

4.6 Problem 6

The solution for this problem is equal to $\frac{1}{2}[sin(2x)sin(2t)]$. Thus it can be interpreted that the pulse is decreased by half while the multiplication of $cos(2t)$ and $cos(2x)$ shows that it is the $cos(2x)$ that is moving to the left and right while $cos(2t)$ moves to the up and down.Therefore, makes it oscillates back and forth.

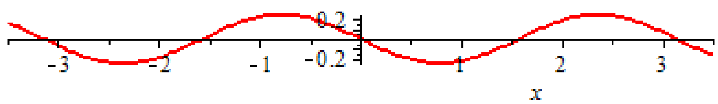

Figure 5: The wave equation solved from problem 6.

5 Conclusion

The d'Alembert formula has an easy way to be derived that is by using simple mathematical calculation including integration and simultaneous equations. This d'Alembert formula is known to be used to solve the standing waves effectively.

There are overall six problems that have been solved manually. As the problems are vary from each other, the different kind of solutions are obtained and the behavior of the waves that is solved by using the d'Alembert formula can be understood deeply.

By using the Maple Software, all of the graphs from six problems that the solutions have been discovered are plotted. After plotting the animation of the graphs,it shows clearly that the wave is moving towards the opposite directions.

For the future research, the recommendation to extend this project is to include other methods that can be used to solve the wave equations problems, to compare it with the current method which is by using the d'Alembert formula. In this study, d'Alembert formula has been used to solve the one-dimensional wave equation. In order to extend the research of this project, the researcher should discover either d'Alembert formula can be used to solve higher dimensional wave equation problems or not.

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The Mathematical Modelling of Shallow Water Wave

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Abstract

In this article, the study have been conducted on a nonlinear waves equation, Korteweg-de Vries (KdV), arising in mathematical modelling and the investigating existence of solutions to these equation using variational methods. In particular, the traveling wave solution is known as solitary waves.The focus is on the derivation of the Korteweg-de Vries equation and the solution for these equations.For methodology, there are two methods that have been used which are exact equation to the Kdv Equation using D'Alembert Method and Exact Solution with Backlund transformation (Bilinearization). Exact equation to the KdV equation using D'Alembert method are leads to two waves represents by $f(x - ct)$ which is a wave that move towards right with speed c and $f(x+ct)$ which is wave that move towards left with speed c. Next, the method is Backlund transformation. By using this method it provides the approach to the theory of solitary wave. The arbitrary function is used to obtain the dependent variable transformation. After that the next step is to get the two soliton solutions by using the mathematica code.The method of Backlund transformation create a non-linear equation presenting the soliton solution.The KdV equation is rewritten in a bilinear form in order to find the Backlund transformation for equation KdV.To obtain the result, maple code are being used. Maple are being used to plot the graph of one soliton and two soliton solution waves. It has been plotted in 2d graph. For the conclusion, there is comparison between D'Alembert method and Backlund Transformation method to determine which method is more easier.

Keywords: KdV equation, D'Alembert Method, Backlund transformation, soliton solutions and mathematica code.

1 Introduction

Waves are one of the most fundamental motions: waves on the water's surface and of earthquakes, waves along springs, light waves, radio waves, sound waves, waves of cloud, waves of crowds, brain waves, and so forth (Toda, 1989).

Waves are recorded and analyzed. In the case of sound waves and light waves, it is customary to analyze a wave as the sum of simple sinusoidal waves. This is the principal of linear superposition.

However, when water waves are being observed carefully, the linear superposition principle cannot be applied in general, except for very small amplitudes. The study of water waves of finite amplitude was one of the main topics.

According to article from Wikipedia (2014), in fluid dynamics, dispersion of water waves generally refers to frequency dispersion, which means that waves of different wavelengths travel at different phase speeds.

Other than that, according to the article in Star online the flood that occurred in Kelantan such as Kuala Krai,Pasir Mas, Tanah Merah and others were the worst flood in 2014.

According to Robinson (2014) when a head storm develop and moves downstream,then a rapid rise of water in a creek was occurs that we called flood wave .These was happened in Kuala Krai and many things were destroyed caused of the flood.

In mathematics, the Korteweg–de Vries equation (KdV equation for short) is a mathematical model of waves on shallow water surfaces. It is particularly famous as the prototypical example of an exactly solvable model, that is, a nonlinear partial differential equation whose solutions can be exactly and precisely specified.

KdV can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation is a topic of active research. The KdV equation was first introduced by Boussinesq (1877) and rediscovered by Diederik Korteweg and Gustav de Vries (1895).

2 Literature Review

According to Toda (1989),he tried to review the study of non-linear waves and present the method of solving the most important non linear wave equations in a compact, clear and coherent way. There are some topics that have been discussed in this book. One of the topic is the KdV equation. In this section, a solitary wave with steady form can exist on shallow water. Hence, consider the waves changing with time, and derive the famous KdV equation.

According to Johnson (1989),the authors write an introduction to the theory of solitons and its diverse applications that arise in the physical sciences of non-linear systems .The Inverse Scattering Transform is explained by the generation and properties of solitons and introducing the mathematical techniques. In this book,the topics covered which are solitary waves,the Korteweg-de Vries equation and its initial-value problem,linear direct and inverse scattering problems,conservation laws,the Lax formulation,Hirota's bilinear method, Backlund transformation,the Painleve property and numerical methods.

Another study conducted by Brauer (2014), the author overall writes about the Korteweg-de Vries equation (KdV) which is a non-linear Partial Differential Equation (PDE) of third order of travelling waves as solutions. In this article, the author's aim is to present an analytical exact result to the KdV equation by means of elementary operations as well as by using Backlund transform. Other than that, the author's aim has been to apply numerical discrete methods to non-linear PDEs.

Based on the study as stated in Matsuno (1984), the author discuss about the single exact solution using Backlund transformation by using Bilinearization. In this chapter, the KdV equation is first considered and solve the subscripts denote partial derivative by referring to Matsuno. Then, the author also suggested that by using maple we can derive the equation to solve the bilinearization.

The fifth order KdV had been discussed in Yacob (2004). The author had studied the method of solution of the fifth order KdV by using the Painleve method.

Based on the research from Thacker (1980) the exact solution related to time dependent motion in parabolic basins. The characteristic feature is that the shorelines is not fixed.It is must be determined as part of the solution and free to move.In general, the motion is oscillatory and has the appropriate small amplitude limit.In these case ,there is a solution for the flood wave in which the parabolic basins are reduces to the flate plane.These solution provides a valuable test to numerical model of inundating storm tides.

The study from Caldwell (2012) the model propagation of wave is the fundamental partial differential equation that defines the motion of linear waves such as light, water and sound of waves.He obtain the same equation as model of vibrating string that is one-dimensional homogenous wave of equation.He simplify the notation by introducing the d'Alembert operator.

3 Methodology

The derivation is begin with the Korteweg-de Vries equation. The derivation of the KdV equation begins by considering the boundary value problem (BVP) of the two-dimensional incompressible, inviscid and irrotational fluid (water) flow of finite depth bounded above by a free and below by a rigid horizontal plane. The motion is governed by Euler's and Laplace equations.

In order to derive, by using transformation variables

$$\xi = \frac{\alpha^{\frac{1}{2}}}{\delta}(x-t) \qquad ,\tau = \frac{\alpha^{\frac{3}{2}}}{\delta}t \qquad z = z \tag{1}$$

to transform coordinate (x, t, z) to coordinate (ξ, τ, z). According to Yacob (2004), Jeffrey and Kakutani stated this new coordinate is more suitable for describing the traveling wave. Yacob (2004) stated that Jeffrey & Kawahara (1982) used this strain coordinate to investigate the behaviour of the fluid.

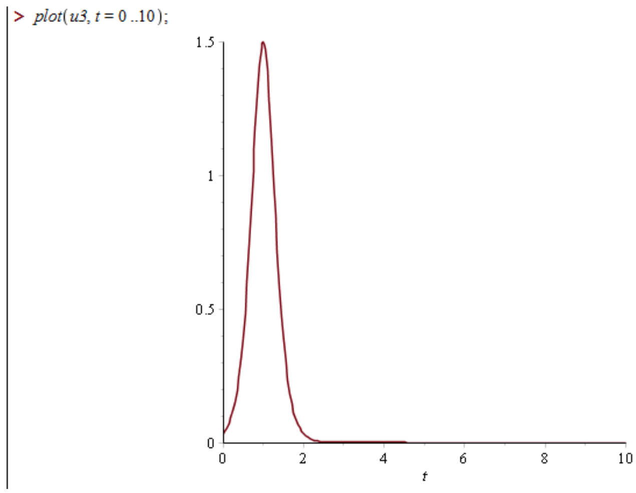

Figure 1: One Soliton wave for D'Alembert

Transformation (1) is known as the Gardner–Morikawa transformation. By using transformation (1) and set $\Phi(\xi, z, \tau) = \alpha^{1/2}\phi(x, z, t)/\delta$, the KdV equation can be derived successful. With help of Maple 18, the derivation of the KdV can be easily implemented.

D'Alembert method has being used in this project by refer to the Brauer (2014), a function of the form $u(x, t) = f(x - ct)$ as the simplest mathematical wave for PDE $u_t + cu_x$.This solution have two wave fronts that represent by term $f(x - ct)$ and $f(x + ct)$.The KdV equation is lead to ODE and being integrate.Next, the boundary condition is being impose $z, \frac{dz}{d\xi}, \frac{d^2z}{d\xi^2} \to 0$ as $\xi \to \pm\infty$ which describe the solitary wave. Finally, the equation that will get is $u(x, t) = \frac{\beta}{2}\,\mathrm{sech}^2[\frac{\sqrt{\beta}}{2}(x - \beta t)]$ by substituting and transforming the equation.

The Backlund Transformation is presented in operating on solution of KdV equation. The method is used to linearize the Kdv equation to linear of PDE. By using Backlund transformation we will vanishing boundary condition as x approach to ∞. Consider that to obtained the KdV equation by using $u = 2\frac{\partial^2}{\partial x^2}(\ln(F))$ and the pertubation method on f with respect to the small parameter ϵ.

With the help of *Mathematica*, the graph and animation of the one soliton and two soliton solutions are obtained.

4 Results And Discussion

For one soliton solution of D'Alembert method the result is obtained by using *Maple*. For Backlund Transformation, we have obtain its result by using *Mathematica*. The graph of 3D KdV of one soliton solution are obtained.

To show the solution of D'Alembert for two soliton for different value of

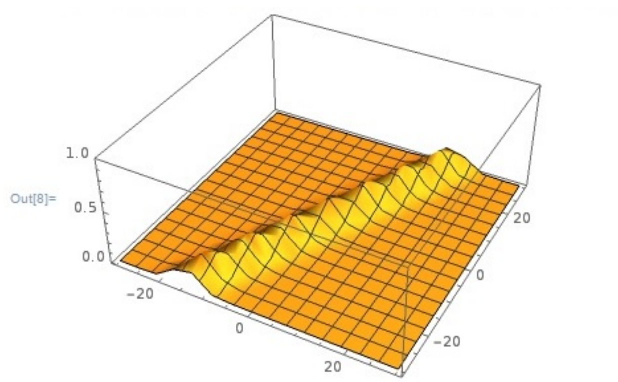

Figure 2: One Soliton using Backlund transformation

time,t, it is possible to generate an array of graphics with type 'GIF' with *Mathematica*.The characteristic snapshots are as follows;

Figure 3: Two soliton Solution for D'Alembert

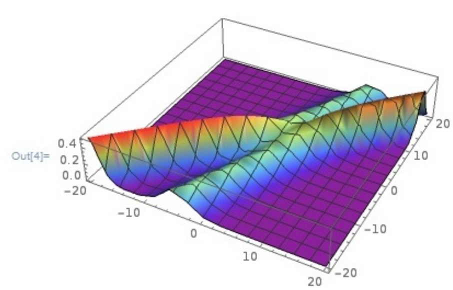

Figure 4: Two Soliton using Backlund transformation

The solitary wave is observed and it shows that the higher amplitude is exactly half the speed.Thus larger solitary waves have greater speeds. The two solitary wave solutions with centers will splitting apart with different amplitude.Based on D'Alembert result graphic we use the equation $u(x,t) = \frac{\beta}{2}\,\text{sech}^2[\frac{\sqrt{\beta}}{2}(x-\beta t)]$ to satisfy the Kdv equation.Meanwhile for the result Backlund Transformation we use the equation $u = 2\frac{\partial^2}{\partial x^2}(\ln(F))$

5 Conclusion

From this project, the study have been conducted on the fundamental of wave equation. The limit of this research research only on shallow water wave that involve with Korteweg-de Vries(KdV) equation. We describe the travelling wave which is $f(x-ct)$ and $f(x+ct)$ in this project.

Based on the objective, the derivation of KdV equation have achieved. In the process of derivation, the Boundary conditions is used on the free surface,$z = \zeta(x,t)$.To make variables in non-dimensionalised form introduce, $\alpha = \frac{a}{h}$, $\delta = \frac{h}{l}$. After that rescaling BVP to get the derivation of KdV.

Moreover the exact solution have found by using D'Alembert method which is $u(x,t) = \frac{\beta}{2}\,\text{sech}^2[\frac{\sqrt[2]{\beta}}{2}(x-\beta t)]$. By using $Mathematica$, the equation returns true. This prove that this equation is the solution to KdV.

Besides that, by using several transformation and fundamental formula, the exact solution is obtained for Kdv equation. By this method it can linearize the KdV equation. The calculation can be simplified by represent the two series of transformation,$u = w_x$ and $w = 2(lnf)_x$. By utilize the superposition principle it will find the two soliton solution since it allocate with a bilinear equation.

For the next study, the solution of higher order $O(\alpha^n)$ of the KdV where $n \geq 1$.

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COMPARING METHOD IN DOUBLE PENDULUM

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Abstract

This paper represent about the motion of the curves in double pendulum by comparing the three types of method that related each other. The method that used in double pendulum are Lagrangian, Euler equation, Hamilton's and lastly Runge Kutta. This method are related each other because to derive the Euler equation, formula of Lagrangian is needed and also from Euler equation, it can derive into two types of method such as Hamilton's and Runge Kutta but Runge Kutta can also derive from Hamilton's. All this method are needed to know their motion, structure of wave, and so on. Mathematica software is needed for solving the problem of double pendulum and to get the accurate result based on graph of parametric and for animation,and also it shows their movements. This software can solve all this method included in Lagrangian.

Keywords: double pendulum, Lagrangian, Euler, Hamilton, Runge Kutta, Mathematica software

1 Introduction

Pendulum that attach with another pendulum is called double pendulum. The area of dynamical system in physic and mathematics, a rich dynamic behavior of a strong sensitivity is exhibits from the double pendulum of simple physic system with initial conditions. Double pendulum have a difference types whether same mass or different mass that declare as m_1 and m_2 and same length or different length that declare as L_1 or L_2. Its also have different angles. In Stickel (2009), a diagram of a double is shown in Fig. 1. The conservative system happens when double pendulum is friction-less that allows a conservation of energy, that is $Energy_{in} = Energy_{out}$. Furthermore, Stickel (2009) mentioned that double pendulum is two masses attached to rigid, mass less, rod with the base at a stationary location . In other words, the double pendulum become a linear system when angle is small and become non linear when angle is big.

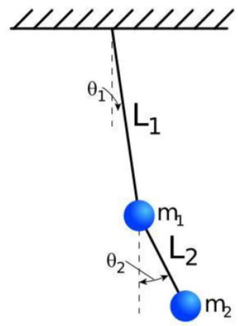

Figure 1: Double Pendulum

To predict the behavior of double pendulum is very limited in certain regimes that is initial condition because the extreme sensitivity towards even small perturbations. In addition, Nielsen & B.T. (2013) said that the double pendulum is considered as a model system exhibiting deterministic chaotic behavior and the motion is governed by a set of coupled differential equations. This project we will use four types of methods to solve the double pendulum which are Lagrangian Equation, Range-Kutta Equation, Hamilton's Equation and lastly Euler Equation. In Stickel (2009), the Lagrangian is representation system of motion and can be used when system is conservative . Determine expressions for the kinetic energy and the potential when apply the Lagrange's equation (S.Widnall, 2009). The general equation for this method is :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1}$$

Runge-Kutta equation is generally to solve differential equation numerically and its very accurate also well behaved for wide range of problems. Generally, the general solution of Runge-Kutta for double pendulum is :-

$$w_0 = \alpha \tag{2}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{3}$$

which is $w_i \approx y(t_i)$ computes an approximate solution. Hamilton's Equation is used when to solve the trajectories of double pendulum. The formula of Hamilton's is:-

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{4}$$

Lastly, Euler-Lagrange equations for θ_1 and θ_2 are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1} \tag{5}$$

All these methods will be discussed in details in methodology.

2 Literature Review

Single Pendulum

A single pendulum has a single particle m hanging from a string of length l and fixed at a point Q. the pendulum will swing back and forth with periodic motion as shown in Figure 2 when displaced to an initial angle and released. As for the simple pendulum, the equation of motion for the pendulum may be obtained by applying Newton's second law for rotational systems.

$$\tau = I\alpha \Rightarrow -mg\,(\sin\,\theta)\,l = ml^2\frac{d^2\theta}{dt^2}$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\,\theta = 0 \tag{6}$$

where τ and α are the force the angular acceleration . If the amplitude of

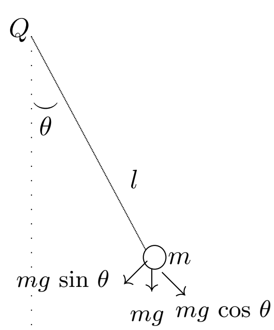

Figure 2: Simple pendulum

angular displacement is small enough that $\sin\,\theta \approx \theta$, equation (6) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{7}$$

gives the simple harmonic solution

$$\theta(t) = \theta_0 \cos(\omega t + \psi) \tag{8}$$

where $\omega = \sqrt{g/l}$ is the natural frequency of the motion. According to Gonzalez (2008), the single pendulum is always at a fixed distance from point and can oscillate in (x, y) plane. Mathematically,

$$v = 0 \tag{9}$$

$$\mid r \mid = l \tag{10}$$

The angular position of the pendulum θ ; which we can use to write:

$$r = l(\sin\theta, \cos\theta, 0). \tag{11}$$

The gravity force of simple pendulum is always in the same direction and same magnitude which is 9.8 N/kg. Both its direction and its magnitude changes as the bob swings to and fro and it always towards the pivot point. Witherden (2001) also mention that the tension pointing towards the origin, along the direction of $-r$ and gravity is along y-direction with gravitational acceleration g:

$$F = T\frac{-r}{\mid r \mid} + mg = -\frac{T}{l}r + mg \tag{12}$$

Notice that there will be two unknown of two equations, T and θ. Notice also the tension force will be greater than the perpendicular component of gravity when the bob moves through this equilibrium position. Kelly (1993) stated that since the bob is in motion along a circular arc, there must be a net force at this position.

Double Pendulum

Double pendulum have a two masses m_1 and m_2 which connected by rigid weightless rod of length l_1 and l_2, subject to gravity forces. According to Gonzalez (2008), each particle moving in the (x, y) plane, and the constraints are holonomic which is they are only algebraic relationship between coordinates but not involving inequalities or derivatives and each rod having constant lengths. Since there are two generalized coordinates, the expression for r_1,r_2 in term of two angles θ_1,θ_2:

$$r_1 = l_1(\sin\theta_1, \cos\theta_2, 0) \tag{13}$$

$$r_2 = r_1 + l_2(\sin\theta_2, \cos\theta_2, 0) \tag{14}$$

The tension in the upper rod is along the direction $-r_1$, and due to the lower rod is along the direction $r_2 - r_1$ and F_1 will be;

$$F_1 = T_1 \frac{-r_1}{\mid r_1 \mid} + T_2 \frac{r_2 - r_1}{\mid r_2 - r_1 \mid} + m_1 g = -\frac{T_1}{l_1} r_1 + \frac{T_2}{l_2}(r_2 - r_1) + m_1 g \tag{15}$$

The tension on $m_2$is along the direction of $-(r_2 - r_1)$ will be:

$$F_2 = T_2 \frac{-(r_2 - r_1)}{\mid r_2 - r_1 \mid} + m_2 g = -\frac{T_2}{l_2}(r_2 - r_1) + m_1 g \tag{16}$$

Witherden (2001) mentioned that since there exists no analytical solution for double pendulum, it must instead be done numerically which there exist several solvers. Considering several methods such as Lagrangian equation which allows for firstly verification and secondly allows for a comparison to be made between methods such as Euler, Hamilton and also Runge-Kutta.

Lagrangian Equation

The Lagrangian can be find by using the equation of the motion of the system in term of generalized coordinates :

$$L = K - P \tag{17}$$

where K represent kinetic equation and P represent potential energy.

As for the conservative system of Lagrangian, Rao & J.Srinivas (2007) stated that Lagrange's equation proposed and approach which will obtain the equation of motion in generalized coordinates of the system from the analytical dynamics points of view which can also be expressed:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{18}$$

The results that describe the equations of motion of the system in the differential equations.

Runge-Kutta Equation

According to Stoer & Bullrsch (1980), there are many ways to evaluate f(x, y), but the higher-order error terms in a different coefficients. Adding up the right combination of these, we can eliminate the error terms order by order. That is the basic idea of the Runge-Kutta method. Lambert (1973) stated that Runge-Kutta 4^{th} order method is a numerical technique used to solve ordinary differential equation of the system.

The fourth order Runge-Kutta method can be expressed as follows:

$$\begin{cases} y' &= f(t,y) \\ y(t_0) &= \alpha \end{cases} \tag{19}$$

By defining h to be the time step size and $t_i = t_0 + ih$. Then, the following formula can be expressed as:

$$w_0 = \alpha \tag{20}$$

$$m_1 = hf(t_i, w_i) \tag{21}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{22}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{23}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{24}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{25}$$

which computes an approximate solution, that is $w_i \approx y(t_i)$.

According to Rice (1983), to achieve some predetermined accuracy in the solution with minimum computational effort is one of the purpose of adaptive step-size control.Thus, some related conserved quantity that can be monitored although the accuracy may be demanded not directly in the solution itself.

Hamilton's Equation

According to Ramegowda (2001), Hamiltonian or Hamiltonian formulation consists of two independent variables which are canonical coordinates and canonical momenta.These two variables come when replaced the n 2^{nd} order differential equations by $2n$ 1^{st} order differential equations for p_i and q_i.

The formula of Hamilton's Equation can be expressed as :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{26}$$

Hamiltonian advantages are that it leads to powerful geometric techniques for studying the properties of dynamical system. It allows for a beautiful expression of the relation between symmetries and conversation law, and it leads to many view that can be viewed as the macroscopic "classical" (Stroup, 2004).

Euler Equation

According to Batchelor (2008), Euler's equation are the equation that written out entirely in term of the principal axes attached to the rigid body. A derivation of the Euler's equations. The torque equation in terms of the frame fixed that related to rigid body fluid is:

$$\left[I_{xx}\frac{d\omega_x}{dt} + (I_{zz} - I_{yy})\omega_y\omega_x\right] = \tau_x \tag{27}$$

$$\left[I_{yy}\frac{d\omega_y}{dt} + (I_{xx} - I_{yy})\omega_z\omega_x\right] = \tau_y \tag{28}$$

$$\left[I_{zz}\frac{d\omega_z}{dt} + (I_{yy} - I_{zz})\omega_x\omega_y\right] = \tau_z \tag{29}$$

Hunter (2004) stated that the incompressible Euler equation for the flow of inviscid, incompressible fluid, describe some of their basic mathematical features, and provide a perspective on their physical applicability.

The incompressible Euler equations are the following PDEs for (il, p) :

$$il_t + il \cdot \nabla il + \nabla p = 0 \tag{30}$$

$$\nabla \cdot il = 0 \tag{31}$$

The Euler-Lagrange equations for θ_1 and θ_2 are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1} \tag{32}$$

The θ_1 equation is :

$$\ell_1\left[(m_1 + m_2)\ell_1\ddot{\theta_1} + m_2\ell_2\cos(\theta_1 - \theta_2)\ddot{\theta_2} + m_2\ell_2\sin(\theta_1 - \theta_2)\dot{\theta_2^2} + (m_1 + m_2)g\sin\theta_1\right] \tag{33}$$

and the θ_2 equation is :

$$m_2\ell_2\left[\ell_2\ddot{\theta_2} + \ell_1\cos(\theta_1 - \theta_2)\ddot{\theta_1} - \ell_1\sin(\theta_1 - \theta_2)\dot{\theta_1^2} + g\sin\theta_2\right] = 0 \tag{34}$$

3 Methodology

3.1 Step 1:Development of Lagrangian Equation for Double Pendulum

First, the x-axis pointing along the horizontal direction and the y-axis pointing vertically upwards and fixed point O will be taken as the origin of the Cartesian coordinate system. Let θ_1 and θ_2 be the angles which the vertical direction of the first and second rods make with respectively.Now, we will consider the Lagrangian equation by :

$$L = K - P \tag{35}$$

where K represents kinetic equation and P represents potential energy.From the potential equation,$P = mga$, we can find the potential energies for the first and second pendulums by simplifying them.In order to implement in the Runge-Kutta equation, first, Lambert (1973) mentioned that we should identify the fourth-order of Runge-Kutta which can be seen as ODE integrator. In order to carry out the Runge-Kutta, we need to input the values of the independent variables on a set of n differential equation and step-size, h. In the end, the solution will be:

$$w_0 = \alpha \tag{36}$$

$$m_1 = hf(t_i, w_i) \tag{37}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{38}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{39}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{40}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{41}$$

which $w_i \approx y(t_i)$ computes an approximate solution. In order to get into Runge-Kutta (RK4), we need to develop Euler-Lagrange and Hamiltonian first.By using software Mathematica, we would generate the equation to get a significant result.

3.2 STEP 2 : Euler-Lagrangian Equation Development from Lagrangian System

The Lagrangian formulation,the function $L(p_i, q_i, t)$ where p_i and q_i $(i = 1, \cdots, n)$ are n generalized coordinates (Kelly, 1993).Euler-Lagrange system is also called

"Lagrange's Equation of Second Kind".The Hamilton's equation can be derive from the Lagrange equation by substitute the value of p_i and q_i. The final solution will be :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{42}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{43}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{44}$$

These are the 2^{nd} order differential equations which require 2n initial conditions. In order to make it n generalized equation, Ramegowda (2001) stated it will be canonical momenta equation.In order to get to the Hamiltonian, we need to develop Euler-Lagrange equation. Software Mathematica could be generated in finding the result of double pendulum problem by this equation. The implementation and the result will be briefly discuss below.

3.3 STEP 3 : The Expansion of Hamiltonian into Runge-Kutta

Suppose that the upper pendulum has a massless rod of length ℓ_1 and a bob of mas m_1. The two rods provide constraints on the motion of the vertical plane which can compute into x_1, x_2, y_1 and y_2. Consider the Lagrangian equation by :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{45}$$

From the above equation, Batchelor (2008) states that the Lagrange equation will be expand to include the Euler equation which consist of two independent generalized coordinates, θ_1 and θ_2.This two angles make the two rods going downward vertical direction repeatedly. By substituting equation of θ_1 and θ_2 in Lagrangian equation, we may have the Euler-Lagrangian equation which is :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1} \tag{46}$$

This type of formula is easily to be computed in the Mathematica software since it is only related between Euler and Lagrange formula.

4 Results And Discussion

4.1 Lagrange

The results of motion of curves by comparing three type of methods in double pendulum when using the fixed data such as m_1=1kg and m_2 =1kg, ℓ_1=2m and ℓ_2=2m, g=9.81n. According to mathematica result, the Figure 4.1 shows the result of Lagrangian equation. The graph in Figure 4.1 shows that the double

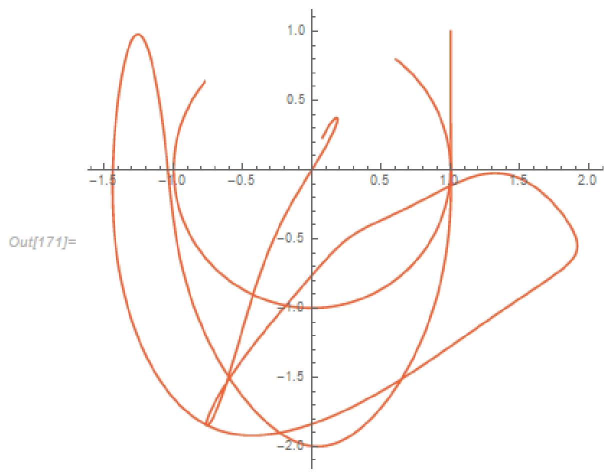

Figure 3: Graph of Lagrangian

pendulum move extremely starting when t=o until the t_max=5sec and its show the motion of curves are dramatically.

4.2 Euler-Lagrange

For the second method, Euler equation, the Figure 4.2 present the motion of curves in double pendulum are more straight upward and shows some curves starting when $t = 0$. The graph also shows, the Euler's method unsuitable for long set time because the t_max=5sec.

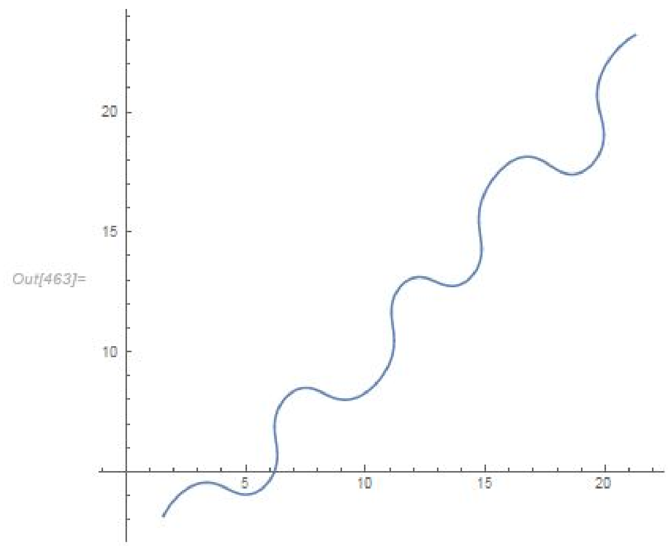

Figure 4: Graph of Euler-Lagrangian

4.3 Runge-Kutta

Figure 4.3, Figure 4.4 and Figure 4.5 represent the result by applying Runge Kutta. The result show the smooth line motion and the curve of the graph can generate the other form. From all this above result for three type of methods it can concluded the Runge Kutta are best method for applying double pendulum.

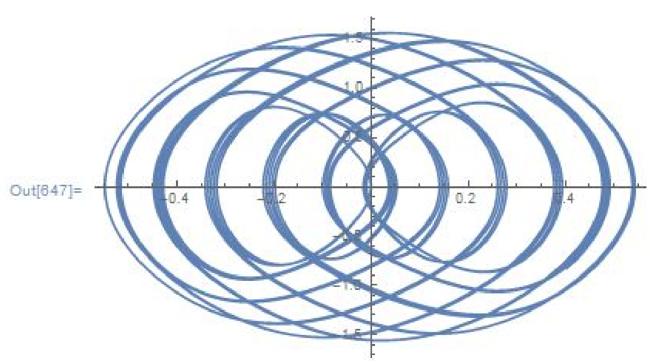

Figure 5: Graph of Runge-Kutta(RK1)

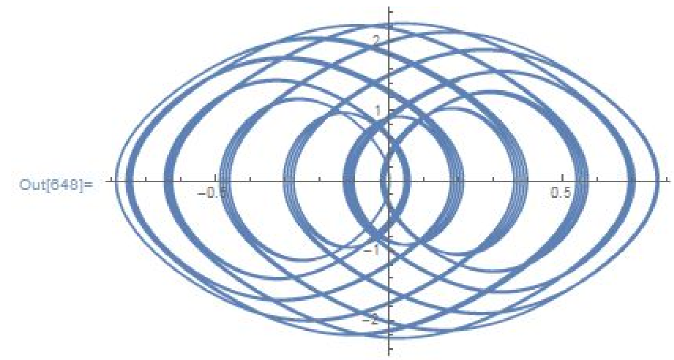

Figure 6: Graph of Runge-Kutta(RK2)

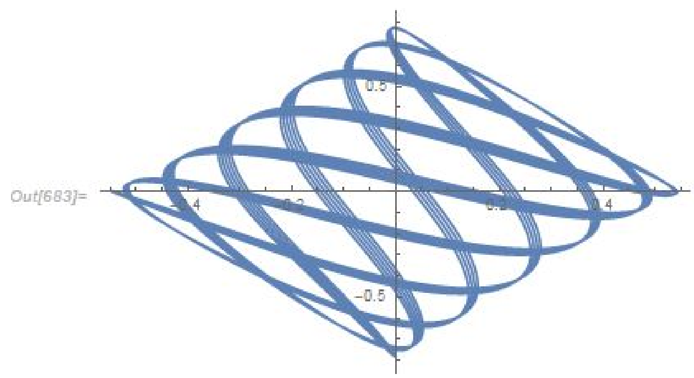

Figure 7: Graph of Runge-Kutta(RK3)

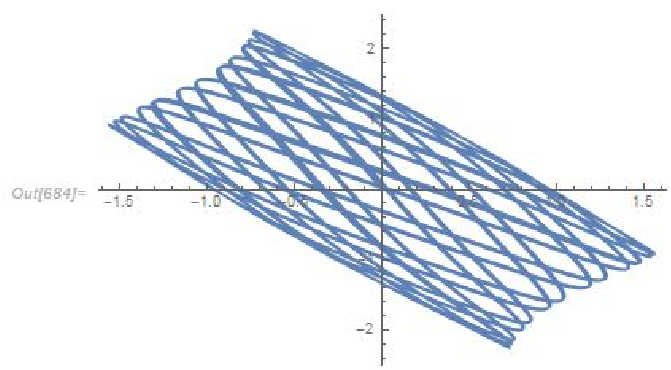

Figure 8: Graph of Runge-Kutta(RK4)

5 Conclusion

The objective are to comparing three type of methods based on their motion of curves. Motion of curves represents the initial condition that sensitive depends and how much chaotic motion. Each of method are related to each other. To knows which one are the best method, Mathematica software has been used to solve this problem. By using the same data and different method which is m_1=1kg and m_2=1kg, ℓ_1=2m and ℓ_2=2m, g=9.81n, the result show that Runge Kutta are the best method other than Lagrangian Equation and Euler's Method because the line of curves are smooth rather than line of curves for other difference method. In addition, the result also shows that Euler's method are not suitable for long time-step. In conclusion, Runge Kutta are the best method to solve double pendulum whether in long time-step or short time-step. Lastly, for the recommendation to get the better result the findings can take a long time when doing an experiment for double pendulum but if other researchers want to get the better result for Euler's method, it can take a short time-step because Euler's method only suitable for short time-step. Furthermore, it also can use difference mass which is m_1¡m_2 or m_1¿m_2.

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